Problem Set 3

Problem 1. Endogenous Fertility

As we argued in class, population in our one-sector growth model is a fixed factor which is not accumulated and thus implies that there can be no growth in the long-run - if we have constant returns to scale production function. While this is a valid argument, perhaps the assumption that population is a fixed factor is not a good assumption. In particular, one can imagine that the decision to have children is one which is actually affected by economic conditions. Here we try to develop this insight and work on a model with endogenous fertility. This is based on the paper by Barro and Becker.

Consider the following intergenerational setting where in each period a new generation of individuals are born. They have the following preferences

$$W_t = u(c_t) + \beta \frac{g(n_t)}{n_t} \int_0^{n_t} W_{t+1}^i di$$

In other words, each generation lives for one period and get utility from their own consumption and the utility of their kids. The function $g(n)/n_t$ captures the degree of altruism and depends on the number of children. Note that we assumed that the number of children is continuous choice. At an aggregate level, this is not a crazy assumption. If we assume that all the children are the same, then we can write

$$W_t = u(c_t) + \beta g(n_t) \int_0^{n_t} W_{t+1}^i di = u(c_t) + \beta g(n_t) n_t W_{t+1}$$

Thus combining all of this, we have

$$W_t = \sum_{s=t}^{\infty} \beta^{s-t} \left(g(n_t) \cdots g(n_{s-1}) \right) u(c_s)$$

The total population is given by

$$N_t = N_0 n_0 \cdots n_{t-1}$$

On the cost side, cost of children is in terms of time. In other words, if an individual decides to have n_t children, this will cost $bn_t w_t$ where w_t is the wage rate in the economy. In other words, every child will lead to a loss of b units of labor supply.

Assume that the individuals have access to capital and inelastically provide labor and that the production function is given by a standard Cobb-Douglass Production function using capital and labor.

A useful notation for this problem is to think about aggregate and per capita variables, i.e., $c_t = \frac{C_t}{N_t}$, $y_t = \frac{Y_t}{N_t}$, etc.

a. What is output in this economy given that population is given by N_t while capital is K_t? Derivate the feasbility constraint.

b. Write the budget constraint for a dynasty of households in this economy. Do this using a sequential market setting.

c. Define a competitive equilibrium for this economy. Note that since there is no heterogeneity, it is without loss of generality to assume that saving is done in the form of capital.

d. Write the planning problem for this economy for a planner that maximizes the welfare of the initial generation. Show that the solution to this is part of a competitive equilibrium defined above. In other words, find prices for which any solution to the planning problem is a CE.

e. Suppose that $u(c) = \frac{c^{1-\sigma}}{1-\sigma}$ with $\sigma > 1$, what do we have to assume about $g(n)$ so that this problem is mathematically well-behaved, i.e., fertility is non-zero?

f. Write the planning problem recursively. What are the state variables?

g. Suppose that $u(c) = \frac{c^{1-\sigma}}{1-\sigma}$ with $\sigma > 0$ and that $g(n) = n^{1+\eta}$ what assumption on η and σ should be made so that people will always have children?

h. Assume the above specification, define a BGP for this economy. Suppose a BGP exists. What is the long-run growth rate of the economy?

i. Suppose that $1 - \sigma = \eta$, calculate the long-run growth rate of the economy. Which of the models that we have discussed in class os similar to this? Answer intuitively or mathematically.

j. For the economy in part i, calculate the population growth rate. How is this correlated with the growth rate of the economy. Discuss this relationship and relate it to what we observe in the data. Are they in line? If not, what is the main issue with the model developed above.

Problem 2. A Model of Vintage Capital In our growth models so far, we have assumed that there is one type of capital. In reality, however, capital has vintages: new generations of capital are different than older ones – think new computers versus the old ones. In this problem, you will formulate what happens when there are many vintages of capital. To do this, consider a continuous time model where time, t, satisfies $t \in [0, \infty)$. There is a unit mass of consumers with utilities given by

$$\int_0^\infty e^{-\rho t} \frac{C_t^{1-\gamma}}{1-\gamma} dt$$

where C_t is total consumption at t. Consumers are endowed with labor which they supply inelastically and they will receive dividends from firms who own their own capitals of different vintages. Dividends are firms' earnings less wage payments and investment.

Firms produce using different kinds of capital and labor according to the production function

$$Y_t = K_t^\alpha N_t^{1-\alpha}$$

where K_t satisfies

$$K_t = \left(\int_{-\infty}^t A_{t-v} \left(z_v k_{v,t} \right)^\sigma dv \right)^{\frac{1}{\sigma}}$$

In the above $k_{v,t}$ is the level of capital of vintage $v \in (-\infty, t]$ at time t. As it is clear, in period t only capitals of vintage $v \leq t$ can be used in production. Moreover, $0 < \sigma < 1$. All vintages of capital depreciate at the same rate δ and therefore, law of motion of vintage v capital is given by

$$\frac{\partial k_{v,t}}{\partial t} = -\delta k_{v,t} + u_{v,t}$$

where $u_{v,t} \geq 0$ is investment in vintage v. For vintages of capital $v \leq 0$, at time 0 there is an initial level given by $k_{v,0}$. For other vintages, firms choose x_t which is the level of vintage t at time t – these are the vintages that become available at t. Firms choose investment and labor to maximize the present value of dividends. We assume that $z_v = e^{\eta v}$ and that $A_s = e^{-\theta s^2}$.

1. Define competitive equilibrium for this economy. Recall that we discussed how to setup the problem when firms are forward looking.

2. Define a BGP for this economy and calculate the economy's growth rate on a BGP.

3. Formulate the planning problem associated with competitive equilibrium. If you were to write this problem recursively, what would be the appropriate state variable?

4. Describe the firms optimal investment strategy in a BGP? Which vintages of capital are invested in and used as the economy grows.

Problem 3. Growth and Climate Change In this problem, we study an economy with use of energy and climate change. Consider an economy comprised of a continuum of consumers who have one unit of time and have the following preferences

$$\int_0^{-\infty} e^{-\rho t} \log C_t dt$$

Production in the economy is done using oil, capital and labor. We assume that oil is in finite supply initially at R_0 and if O_t is the level of oil used in production in period t,

$$\dot{R}_t = -O_t$$

where $\dot{R}_t$ is the derivative of total oil resources at time t. We assume that using oil for production creates Carbon footprint and the stock of Carbon affect production. In particular, the total stock of carbon is given by

$$\dot{S}_t = -\gamma S_t + O_t$$

with the initial level of carbon given by S_0 where $0 < \gamma < \rho$. Given the stock of carbon, output in period t is given by

$$Y_t = S_t^{-\sigma} O_t^{\alpha} K_t^{\beta} N_t^{1-\alpha-\beta}$$

with $0 < \alpha, \beta < 1$, $\alpha + \beta < 1$ and $\sigma > 0$.

1. Define a competitive equilibrium for this economy where capital, labor and consumption are traded.

2. Write the planning problem associated with this economy.

3. In the planning problem, define a BGP and calculate the long-run growth rate of this economy – assume that a BGP exists.

4. Intuitively, describe what policies in the competitive equilibrium of this economy will acheive efficiency.

Problem 4. Semi-Endogenous Growth

Solve Exercise 13.22 in Acemoglu's textbook.

Problem 5. A Two Country Model of Growth

Consider the standard one-sector growth model in continuous time and assume that the world economy is consisted of two countries. Suppose that the two countries only differ in their Total Factor Productivity. Furthermore, assume that capital and consumption goods flow freely across the two countries but labor does not. Suppose that the representative agent in each country has preferences given by

$$\int_0^\infty e^{-\rho t} u\left(c_t\right) dt$$

and that there are a unit continuum of consumers in each country.

a. Define a Competitive Equilibrium. Either assume time-0 trading or sequential markets.

b. We can apply FWT to show that the competitive equilibrium defined above is Pareto Optimal. Write down the optimization problem associated with Pareto optimality.

c. Write down the pareto problem recursively. How many state variables do you need?

d. Characterize the steady state of this model.

e. Now suppose that we shut down global flow of capital and goods. Characterize the steady state of the model. How does your answer differ from part d. Provide an intuition for the difference.

f. Suppose now at time 0 a shock destorys country 1's capital by 50%. Describe what happens as we transition to the steady state. How does world income inequality evolve over time? Is there a gradual convergence?

Problem 6. A Ricardian Model of Trade and Growth

The previous exercise, while informative, does not tell us anything about how growth and trade shoud be correlated. In this problem, you will develop a model that allows you to answer this question. We will do this in the context of the AK model. But before going dynamic, let's start thinking about trade - we have not really discussed trade before since we need to allow for production and trade of multiple commodities.

A Ricardian Model of Trade. To do this, let us start from a simple static economy where there are two countries, $j = 1, 2$, each with a unit continuum of households each of which have endownment K_j units of capital. On the production side, there are two types of goods: final goods and intermediate good. There are a continuum of intermediate goods that use capital in their production. Each intermediate good is represented by $\omega \in \Omega = [0, 1]$ and its production function in country j is given by

$$y_j(\omega) = A_j(\omega) k_j(\omega)$$

where $A_j(\omega)$ is varying across goods and $k_j(\omega)$ is the total capital used in production of ω in country j. We assume that

$$A_1(\omega) = z_1\omega$$
$$A_2(\omega) = z_2(1 - \omega)$$

with $z_1 > z_2$. Thus country 1 is relatively better at producing goods with high ω and country 2 is better at producing goods with low ω. Thus ω determines the extent of comparative advantage of the countries. On the other hand, z_j determines the extent of the absolute advantage of a country.

Final goods are produced using intermediate goods and according to the production function

$$Y_j = e^{\int_0^1 \log x_j(\omega) d\omega}$$

Finally, households only consume final goods and earn income from their ownership of capital. Their preferences are therefore simply given by C_j where C_j is the total consumption of the final good in country j.

a. Suppose that the two countries are in autarky. Define a competitive equilibrium.

b. Solve for prices, $p_j(\omega)$, rental rate of capital r_j, as well as quantities of each good produced.

c. Now suppose that intermediate goods are traded across the countries - but capital and final goods are not. Define a competitive equilibrium.

d. Show that with free trade, there must exists a cutoff ω^* such that goods with $\omega > \omega^*$ are produced in country 1 while other goods are produced in country 2. Given this property, characterize prices $p(\omega)$, rental rate of capital and quantities of goods produced in each country.

Next we turn to a growth model that nests the above model of trade. In particular, suppose that time is continuous $t \in [0, \infty)$ and that there is a unit continuum of consumers in each country that have preferences given by

$$\int_0^\infty e^{-\rho t} \frac{c_t^{1-\sigma}}{1-\sigma} dt$$

Suppose production in each country is the same as above. The assumption is that of all the final goods produced in each country are either consume by households or invested in capital which is then used for production in the future.

e. Suppose that each country is in autarky. Define a C.E. for this economy.

f. Characterize the BGP of the model. What is the growth rate of each country?

g. Now suppose that countries engage in free trade of intermediate inputs - and as before, capital and final goods are not traded. Define a C.E. for this economy.

h. Use the same analysis as in the static model to characterize the set of intermediate goods that are produced in each country in a balanced growth path.

i. Solve for the growth rate of the two economies in a BGP and compare to your answer to part f. Provide intuition for the difference.